Name:________________________________

MA 8663 Section 01 — *Practice Exam 1* — *November 19, 2019*

Follow the instructions for each question and show enough of your work so that I can follow your thought process. If I can't read your work, answer or there is no justification to a solution you will receive little or no credit!

1. Let $\mathcal{H}$ be a Hilbert space and $\{x_k\} \in \mathcal{H}$ be an orthonormal system. Prove that Parseval's identity holds for $\{x_k\}$ if and only if $\{x_k\}$ is complete.

2. Let X be a separable Hilbert space with ON basis $\{x_k\}$. Let $A : X \to X$ be a bounded linear operator. Prove that

$$\lim_{k\to\infty} |\langle Ax, x_k\rangle| = 0$$

3. Let $\mathbb{D}$ be the unit disk in $\mathbb{C}$. Define the following space

$$H^2(\mathbb{D}) = \{f(z) \text{ is holomorphic on } \mathbb{D} : \|f\| < \infty\}$$

where

$$\|f\|^2 = \lim_{r\to 1} \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^2 \, d\theta$$

and $z = re^{i\theta}$. Prove that $H^2(\mathbb{D})$ is a Banach space. Be sure to show $\|\cdot\|$ actually defines a norm. Is it a Hilbert space?

4. Let $M_{mn}(\mathbb{C})$ be the set of all $m \times n$ matrices with complex entries. Define the map $\langle\cdot,\cdot\rangle : M_{mn}(\mathbb{C}) \times M_{mn}(\mathbb{C}) \to \mathbb{C}$ by $\langle A, B\rangle = \operatorname{tr}(B^*A)$. Prove that $\langle\cdot,\cdot\rangle$ defines an inner product on $M_{mn}(\mathbb{C})$. Moreover prove that $M_{mn}(\mathbb{C})$ is a Hilbert space with this inner product.

5. Let $f \in L^2[0,1]$ and define the following operator $T : L^2[0,1] \to L^2[0,1]$ by

$$(Tf)(x) = \int_0^x f(t)\, dt\,.$$

Prove that T is a bounded operator and $\|T\| = 2/\pi$. Compute Ker T and Ran T^*. Is is true that Ran $T^* \neq (\text{Ker } T)^{\perp}$?

6. Let $f \in L^2[a,b]$ and $g \in L^\infty[a,b]$ and define the following operator $M_g : L^2[a,b] \to L^2[a,b]$ by

$$(M_g f)(x) = g(x)f(x)$$

Prove that M_g is a bounded operator and $\|M_g\| = \|g\|_\infty$.

7. Let $x \in \ell^2(\mathbb{N})$. Define the following operator $A : \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N})$ by:

$$Ax = \left\{k^2(x_{k+1} - x_k)\right\}_{k=1}^{\infty}$$

1.) Show that A is an unbounded operator and compute Ker A.

2.) If $x_1 = 0$, compute A^{-1}.

3.) What condition is needed to guarantee that A^{-1} is bounded.

8. Define the following operator on $L^2[0, 1]$:

$$A = i\frac{d}{dx}$$

Show that A is an unbounded operator and compute Ker A. Be sure to have a proper domain for A.

9. Let X be a normed vector space and $T : X \to X$ be a bounded linear operator such that $\|T\| < 1$. Prove that $I - T$ is invertible and that

$$(I - T)^{-1} = \sum_{k=0}^{\infty} T^k \text{ and } \left\|(I - T)^{-1}\right\| \leq \frac{1}{1 - \|T\|}.$$